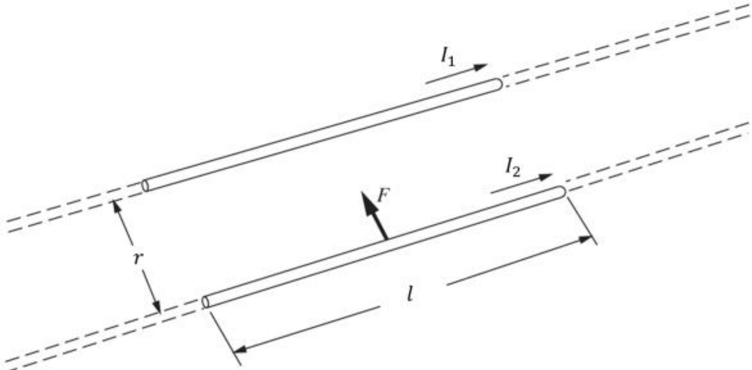
## —Chapter 6—

# The Magnetic Field and Vector Potential

# 6-1 Magnetic Field and Ampère's Law

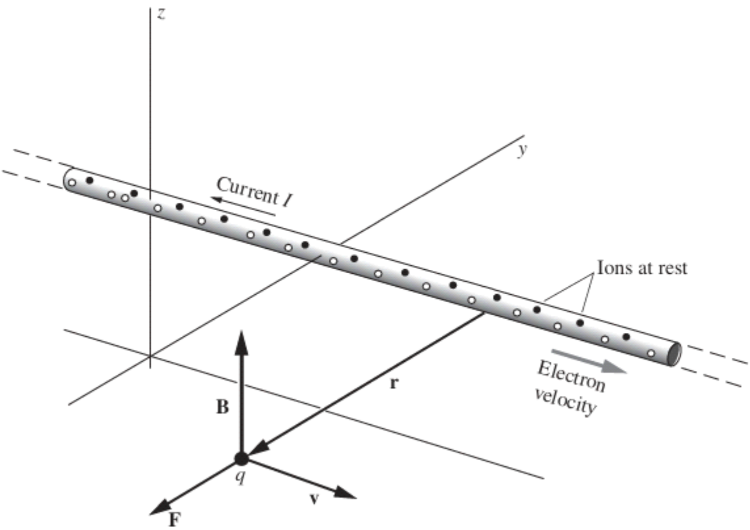
## A. LORENTZ FORCE

(1) Observing two wires running parallel to one another and carrying currents in the same direction




The two sections of wire tend to fly together. The force which depends only on the charge movement in the wires, that is, on the two currents are called *magnetic*.

(2) Observing the motion of a free charged particle, instead of a wire carrying current




In the lab frame, the electric field $\vec{E} = 0$. With the charge $q$ moving with velocity $v$ in the potisitive $x$ direction, the force, according to Coulomb's law and special relativity, on it was in the negative $y$ direction, with magnitude:

$$\vec{F} = -\frac{\mu_0 q v I}{2\pi r}\hat{y}$$

Since an electric current has associated with it a magnetic field that pervades the surrounding space. Any moving charged particle that finds itself in this field, experiences a force proportional to the strength of the magnetic field in that locality. Since the force is perpendicular to the velocity, $\vec{v} \perp (-\hat{y})$, we found that

$$\hat{x} \times \hat{z} = -\hat{y}$$

Thus, we have

$$\vec{v} = v\hat{x}$$

$$\vec{B} = B\hat{z}$$

and can obtain the force

$$\vec{F} = qv\left(\frac{\mu_0 I}{2\pi r}\right)(-\hat{y}) = qv\hat{x} \times \frac{\mu_0 I}{2\pi r}\hat{z} = q\vec{v} \times \vec{B}$$

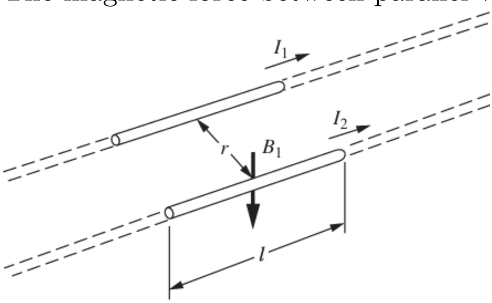
where

$$\vec{B} = \frac{\mu_0 I}{2\pi r}\hat{z}$$

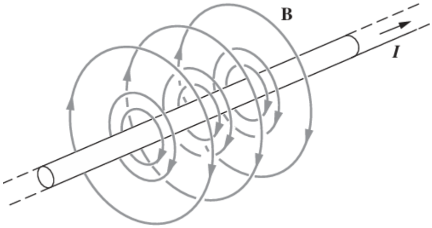
We shall take $\vec{F} = q\vec{v} \times \vec{B}$ as the definition of $\vec{B}$.

EXAMPLES:

1. The magnetic force between parallel wires carrying current.




The magnetic field of a straight wire of filament of steady current:




The field lines are circles surrounding the filament.

$$B_1 = \frac{\mu_0 I_1}{2\pi r}$$

The field direction is everywhere perpendicular to the plane containing the filament. The magnitude of the field is proportional to $1/r$.

Current $I_1$ produces magnetic field $B_1$ at conductor 2. The force on a length $l$ of conductor 2 is given by

$$F = q_2 v_2 B_1 = \frac{q_2 l}{\Delta t} B_1 = I_2 B_1 l = \frac{\mu_0 I_1 I_2 l}{2\pi r}$$

(3) Since boosted electric fields must give rise to magnetic fields. Thus, in general, a charged particle, in our frame, moving with velocity $\vec{v}$, the force on the particle is

$$\vec{F} = q\vec{E} + q\vec{v} \times \vec{B}$$

The force is called the Lorentz force.

EXAMPLES:

1. Show that the magnetic forces do no work.

   ANSWER:

   A charged particle $q$ moves an amount $d\vec{l} = \vec{v}dt$ in the presence of a magnetic field $\vec{B}$. The work done is

   $$dW = \vec{F} \cdot d\vec{l} = q\left(\vec{v} \times \vec{B}\right) \cdot \vec{v}dt$$

   Since $\vec{v} \cdot \left(\vec{v} \times \vec{B}\right) = 0$, we obtain $dW = 0$

## B. AMPÈRE'S LAW

(1) Since the $\vec{B}$ lines curling clockwise, we can look at the line integral of $\vec{B}$ around a closed path in this field.

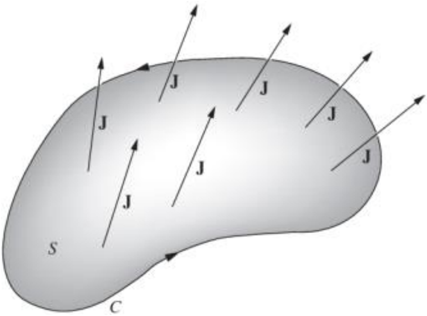
$$\oint_C \vec{B} \cdot d\vec{s} = \mu_0 \times \left(\text{current enclosed by path}\right) \cdots\cdots \text{Ampère's Law}$$

It is valid for *steady* current. Ampère's law states that the line integral of the magnetic field around a **closed loop** is proportional to the current passing through the loop.

NOTE:

We can't apply Ampère's law to isolated finite segments of current-carrying wire. Such finite segments can't exist by themselves, and Ampère's law relies on the full circuit to work. Without the full circuit, by choosing different surfaces bounded by the amperian loop, you can get contradictory results.

(2) Take any closed curve $\mathcal{C}$ in a region where currents are flowing.




The total current enclosed by $\mathcal{C}$ is the flux through the surface spanning $\mathcal{C}$:

$$\oint_{\mathcal{C}} \vec{B} \cdot d\vec{s} = \mu_0 \int_{\mathcal{S}} \vec{J} \cdot d\vec{a}$$

Using Stokes' theorem:

$$\oint_{\mathcal{C}} \vec{B} \cdot d\vec{s} = \int_{\mathcal{S}} \left(\nabla \times \vec{B}\right) \cdot d\vec{a}$$

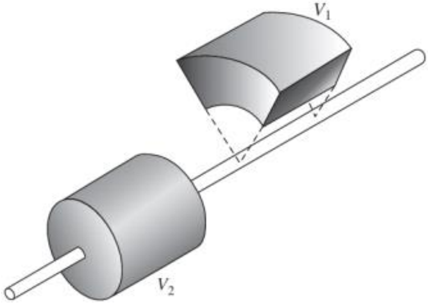
Thus, we obtain

$$\nabla \times \vec{B} = \mu_0 \vec{J}$$

(3) For a given $\vec{J}$, the magnetic field $\vec{B}$ is not uniquely determined by

$$\nabla \times \vec{B} = \mu_0 \vec{J}$$

According to the Helmholtz theorem, we need another condition: we can look at the surface integral of $\vec{B}$ around a closed area in this field.




It is enough to note that the volumes $\mathcal{V}_1$ and $\mathcal{V}_2$ have no net flux. Then, using Gauss's divergence theorem, we get

$$\oiint_{\mathcal{S}} \vec{B} \cdot d\vec{a} = \iiint_{\mathcal{V}} \nabla \cdot \vec{B}\, d\tau = 0 \Rightarrow \nabla \cdot \vec{B} = 0$$

Thus, as $\vec{B}$ goes to zero at infinitely, we have

$\nabla \times \vec{B} = \mu_0 \vec{J}$ ····· Ampère's law

$\nabla \cdot \vec{B} = 0$ ····· no name

and $\vec{B}$ is uniquely determined if $\vec{J}$ is given.

PROOF:

Suppose both equations are satisfied by two different fields $\vec{B}_1$ and $\vec{B}_2$.

$$\vec{D} = \vec{B}_1 - \vec{B}_2$$

$$\nabla \times \vec{D} = \nabla \times \vec{B}_1 - \nabla \times \vec{B}_2 = \mu_0 \vec{J} - \mu_0 \vec{J} = 0 \Rightarrow \vec{D} = \nabla f$$

$$\nabla \cdot \vec{D} = \nabla \cdot \vec{B}_1 - \nabla \cdot \vec{B}_2 = 0 \Rightarrow \nabla \cdot \nabla f = \nabla^2 f = 0$$

Since $\vec{B}_1$ and $\vec{B}_2$ go to zero at infinity (boundary), and

$$\vec{D} = \vec{B}_1 - \vec{B}_2 = \nabla f = 0$$

at the boundary. Thus, $f$ takes on some constant value $f_0$ at the boundary. Since Laplace's equation allows no local maxima or minima—all extrema occur on the boundaries. So $f$ must be the value $f_0$ everywhere. Hence

$$\vec{D} = \nabla f = 0 \text{ and } \vec{B}_1 = \vec{B}_2$$

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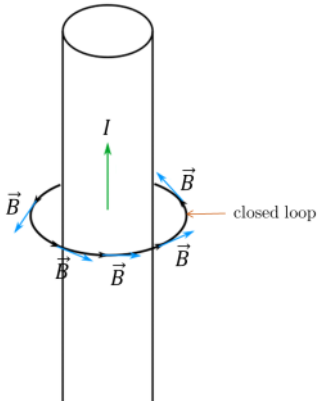
EXAMPLES:

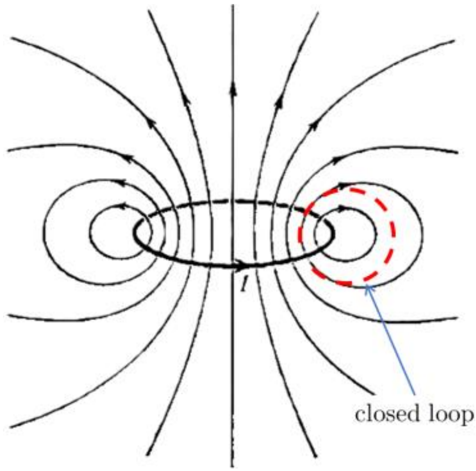
1. Use Ampère's law to find the magnetic field of a long straight wire.




ANSWER:

$$\oint_C \vec{B} \cdot d\vec{s} = \mathrm{B} \oint_C ds = B_\phi \cdot 2\pi r = \mu_0 I \Rightarrow \vec{B} = \frac{\mu_0 I}{2\pi r} \hat{\phi}$$

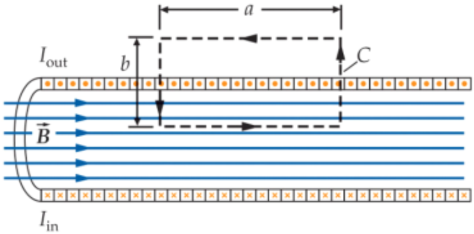
2. Use Ampère's law to find the magnetic field of a ring.




ANSWER:

For Ampère's law to be applicable, the magnetic field strength should be tangent to the loop or normal to the loop. The magnetic field of a ring changes magnitude as well as direction relative to the loop. In this case, there would be no way to take the $\vec{B}$ term out of the line integral, and then Ampère's law cannot be used to determine the field along the loop.

3. Use Ampère's law to find the magnetic field inside a long solenoid.

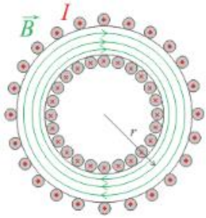



ANSWER:

The magnetic field is zero outside the solenoid and constant inside the solenoid. Thus, along the rectangular loop, we obtain

$$\oint_C \vec{B} \cdot d\vec{s} = Ba = \mu_0 nIa \Rightarrow B = \mu_0 nI$$

4. Finding the magnetic field inside a toroid.




ANSWER:

Cylindrical symmetry ensures that $\vec{B}$ has only $\phi$-component and is constant along any circular path about the axis of the toroidal.

$$\oint_C \vec{B} \cdot d\vec{s} = B_\phi \cdot 2\pi r = \mu_0 NI \Rightarrow \vec{B} = \frac{\mu_0 NI}{2\pi r}\hat{\phi}$$

# 6-2 Vector Potential

## A. ELECTROSTATIC FIELD

(1) Recall that, for an electrostatic field, we have

$$\nabla\cdot\vec{E}=\frac{\rho}{\epsilon_0}$$
$$\nabla\times\vec{E}=0$$

and $\vec{E}$ goes to zero at infinity. According to Helmholtz theorem, if $\rho(r)$ is given, the electric field $\vec{E}$ is

$$\vec{E}=-\nabla\varphi+\nabla\times\vec{A}$$

where

$$\varphi=\frac{1}{4\pi}\int\frac{\nabla\cdot\vec{E}}{|\vec{r}-\vec{r}'|}d^3r'=\frac{1}{4\pi\epsilon_0}\int\frac{\rho(r')}{|\vec{r}-\vec{r}'|}d^3r'$$
$$\vec{A}=\frac{1}{4\pi}\int\frac{\nabla\times\vec{E}}{|\vec{r}-\vec{r}'|}d^3r'=0$$

Thus, $\vec{E}$ is uniquely determined by

$$\vec{E}=-\nabla\varphi$$

where $\varphi$ is called the potential.

(2) Except using the direct integral to obtain the potential, we can also solve Poisson's equation or Laplace's equation with suitable boundary conditions:

$$\nabla\cdot\vec{E}=\nabla\cdot\left(-\nabla\varphi\right)=-\nabla^2\varphi=\frac{\rho}{\epsilon_0}$$
$$\begin{cases}\nabla^2\varphi=-\dfrac{\rho}{\epsilon_0}\\ \nabla^2\varphi=0\end{cases}$$

## B. MAGNETOSTATIC FIELD

(1) For a steady current, we have

$$\nabla\cdot\vec{B}=0$$
$$\nabla\times\vec{B}=\mu_0\vec{J}$$

and $\vec{B}$ goes to zero at infinity. Comparison to the electrostatic field, according to Helmholtz theorem, if $\vec{J}(\vec{r})$ is given, the magnetic field $\vec{B}$ iI

$$\vec{B} = -\nabla\varphi + \nabla \times \vec{A}$$

where

$$\varphi = \frac{1}{4\pi}\int \frac{\nabla \cdot \vec{B}}{|\vec{r}-\vec{r}'|} d^3r' = 0$$

$$\vec{A} = \frac{1}{4\pi}\int \frac{\nabla \times \vec{B}}{|\vec{r}-\vec{r}'|} d^3r' = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|} d^3r'$$

Thus, $\vec{B}$ is uniquely determined by

$$\vec{B} = \nabla \times \vec{A}$$

where $\vec{A}$ is called the vector potential.

(2) Physical interpretation of $\vec{A}$

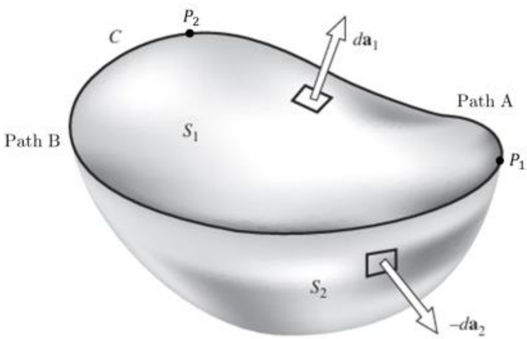
Using Gauss's divergence theorem, we obtain the flux of the magnetic field $\vec{B}$

$$\oiint_{\mathcal{S}} \vec{B} \cdot d\vec{a} = \iiint_{\mathcal{V}} \left(\nabla \cdot \vec{B}\right) d\tau$$

Since the magnetostatic field is a solenoidal field, i.e., $\nabla \cdot \vec{B} = 0$, the total flux through the closed surface is zero. Thus, we have

$$\oiint_{\mathcal{S}} \vec{B} \cdot d\vec{a} = 0$$

Consider two surfaces $\mathcal{S}_1$ and $\mathcal{S}_2$:




$$\iint_{\mathcal{S}_1} \vec{B} \cdot d\vec{a} + \iint_{\mathcal{S}_2} \vec{B} \cdot d\vec{a} = 0$$

$$\iint_{\mathcal{S}_1} \vec{B} \cdot d\vec{a}_1 - \iint_{\mathcal{S}_2} \vec{B} \cdot d\vec{a}_2 = 0$$

Thus, the surface integral is independent of the surface spanned by a closed loop $\mathcal{C}$,

$$\iint_{\mathcal{S}} \vec{B} \cdot d\vec{a} = \underbrace{\int_{P_1}^{P_2} \vec{A} \cdot d\vec{s}}_{\text{path A}} - \underbrace{\int_{P_1}^{P_2} \vec{A} \cdot d\vec{s}}_{\text{path B}}$$

Since

$$\oint_{\mathcal{C}} \vec{A} \cdot d\vec{s} = \underbrace{\int_{P_1}^{P_2} \vec{A} \cdot d\vec{s}}_{\text{path A}} - \underbrace{\int_{P_1}^{P_2} \vec{A} \cdot d\vec{s}}_{\text{path B}} = \iint_{\mathcal{S}} \left(\nabla \times \vec{A}\right) \cdot d\vec{a}$$

thus, we obtain

$$\vec{B} = \nabla \times \vec{A}$$

(3) Poisson's equation or Laplace's equation for $\vec{A}$

$$\nabla \times \vec{B} = \nabla \times \nabla \times \vec{A} = \nabla\left(\nabla \cdot \vec{A}\right) - \nabla^2 \vec{A} = \mu_0 \vec{J}$$

Since

$$\begin{aligned}
\nabla \cdot \vec{A} &= \frac{\mu_0}{4\pi} \int \left(\nabla_r \frac{1}{\left|\vec{r} - \vec{r}'\right|}\right) \cdot \vec{J}\left(\vec{r}'\right) d^3 r' \\
&= \frac{\mu_0}{4\pi} \int \left(-\nabla_{r'} \frac{1}{\left|\vec{r} - \vec{r}'\right|}\right) \cdot \vec{J}\left(\vec{r}'\right) d^3 r' \\
&= -\frac{\mu_0}{4\pi} \oint_{\mathcal{S}} \frac{\vec{J}\left(\vec{r}'\right)}{\left|\vec{r} - \vec{r}'\right|} \cdot d\vec{a} + \frac{\mu_0}{4\pi} \int \frac{1}{\left|\vec{r} - \vec{r}'\right|} \underbrace{\nabla_{r'} \cdot \vec{J}\left(\vec{r}'\right)}_{=0} d^3 r' \\
&= -\frac{\mu_0}{4\pi} \oint_{\mathcal{S}} \frac{\vec{J}\left(\vec{r}'\right)}{\left|\vec{r} - \vec{r}'\right|} \cdot d\vec{a}
\end{aligned}$$

assume that the Gaussian surface is at infinity and the current $\vec{J}\left(\vec{r}\right)$ goes to zero faster than $1/r^2$ as $r \to \infty$. So the surface integral is zero. Thus, we have

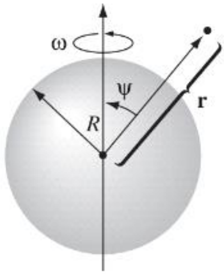
$$\nabla \cdot \vec{A} = 0$$

and obtain

$$\begin{cases} \nabla^2 \vec{A} = -\mu_0 \vec{J} \\ \nabla^2 \vec{A} = 0 \end{cases}$$

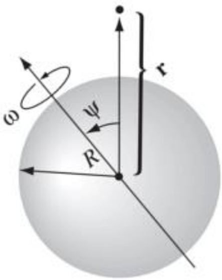
EXAMPLES:

1. A spherical shell of radius $R$, carrying a uniform surface charge $\sigma$, is set spinning at angular velocity $\omega$. Find the vector potential it produces at point $r$.

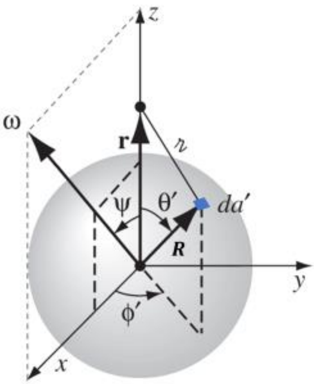



ANSWER:

The integration is easier if we let $r$ lie on the $z$ axis, so that $\omega$ is tilted at an angle $\psi$.




We now let $\omega$ lie in the $xz$ plane.




$$\vec{A} = \frac{\mu_0}{4\pi} \int \frac{\vec{K}(\vec{r}\,')}{\mathscr{r}} da'$$

where $\vec{K}$ is the surface current and $\mathscr{r} = \left|\vec{r} - \vec{R}\right|$,

$$\mathscr{r} = \sqrt{R^2 + r^2 - 2Rr\cos\theta'}$$
$$da' = R^2 \sin\theta' \, d\theta' d\phi'$$

$$\begin{aligned}\vec{K} &= \sigma\vec{v} \\ &= \sigma\vec{\omega}\times\vec{R} \\ &= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \omega\sin\psi & 0 & \omega\cos\psi \\ R\sin\theta'\cos\phi' & R\sin\theta'\sin\phi' & R\cos\theta' \end{vmatrix}\end{aligned}$$

Since

$$\int_0^{2\pi}\sin\phi'\,d\phi' = \int_0^{2\pi}\cos\phi'\,d\phi' = 0$$

the terms involving either $\sin\phi'$ or $\cos\phi'$ contribute nothing. Thus, we can simplify $\vec{K}$ as,

$$\vec{K} = \sigma\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \omega\sin\psi & 0 & \omega\cos\psi \\ 0 & 0 & R\cos\theta' \end{vmatrix} = -\sigma R\omega\sin\psi\cos\theta'\,\hat{y}$$

We obtain the vector potential as

$$\begin{aligned}\vec{A} &= \frac{\mu_0}{4\pi}\int\frac{-\sigma R\omega\sin\psi\cos\theta'\,\hat{y}}{\sqrt{R^2+r^2-2Rr\cos\theta'}}R^2\sin\theta'\,d\theta' d\phi' \\ &= -\frac{\mu_0\sigma R^3\omega\sin\psi}{4\pi}\int\frac{\cos\theta'}{\sqrt{R^2+r^2-2Rr\cos\theta'}}d(\cos\theta')d\phi'\hat{y} \\ &= -\frac{\mu_0\sigma R^3\omega\sin\psi}{2}\int_0^{\pi}\frac{\cos\theta'}{\sqrt{R^2+r^2-2Rr\cos\theta'}}d(\cos\theta')\hat{y}\end{aligned}$$

Letting $u=\cos\theta'$, the integral becomes

$$\vec{A} = -\frac{\mu_0\sigma R^3\omega\sin\psi}{2}\int_{-1}^{1}\frac{u}{\sqrt{R^2+r^2-2Rru}}du\hat{y}$$

Since

$$\begin{aligned}&\int_{-1}^{1}\frac{u}{\sqrt{R^2+r^2-2Rru}}du \\ &\quad= \frac{\left(R^2+r^2-Rr\right)(R+r)-\left(R^2+r^2+Rr\right)|R-r|}{3R^2r^2} \\ &\quad= \begin{cases}\dfrac{2r}{3R^2}, & r\le R \\ \dfrac{2R}{3r^2}, & r\ge R\end{cases}\end{aligned}$$

we have, finally,

$$\vec{A}=\begin{cases}-\dfrac{\mu_0\sigma R^3\omega\sin\psi}{2}\dfrac{2r}{3R^2}\hat{y}=-\dfrac{\mu_0\sigma R\omega}{3}r\sin\psi\,\hat{y}, & r\le R\\ -\dfrac{\mu_0\sigma R^3\omega\sin\psi}{2}\dfrac{2R}{3r^2}\hat{y}=-\dfrac{\mu_0\sigma R^4}{3}\dfrac{\omega\sin\psi}{r^2}\hat{y}, & r\ge R\end{cases}$$

Since $\vec{\omega}\times\vec{r}=-\omega r\sin\psi\,\hat{y}$, we have

$$\vec{A}=\begin{cases}\dfrac{\mu_0\sigma R}{3}(\vec{\omega}\times\vec{r}), & r\le R\\ \dfrac{\mu_0\sigma R^4}{3r^3}(\vec{\omega}\times\vec{r}), & r\ge R\end{cases}$$

Now, we would like to revert to the "natural" coordinates in which $\vec{\omega}$ coincides with the $z$ axis and the point $r$ is at $(r,\theta,\phi)$. We then transform to the coordinates in which $\vec{\omega}$ coincides with the $z$ axis:

$$\vec{A}=\begin{cases}\dfrac{\mu_0\sigma R\omega}{3}r\sin\theta\,\hat{\phi}, & r\le R\\ \dfrac{\mu_0\sigma R^4\omega}{3}\dfrac{\sin\theta}{r^2}\hat{\phi}, & r\ge R\end{cases}$$

We obtain the magnetic field inside the spherical shell:

$$\begin{aligned}\vec{B}&=\nabla\times\vec{A}\\&=\begin{vmatrix}\dfrac{1}{r^2\sin\theta}\hat{r} & \dfrac{1}{r\sin\theta}\hat{\theta} & \dfrac{1}{r}\hat{\phi}\\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial\theta} & \dfrac{\partial}{\partial\phi}\\ 0 & 0 & r\sin\theta\dfrac{\mu_0\sigma R\omega}{3}r\sin\theta\end{vmatrix}\\&=\frac{\mu_0\sigma R\omega}{3}\begin{vmatrix}\dfrac{1}{r^2\sin\theta}\hat{r} & \dfrac{1}{r\sin\theta}\hat{\theta} & \dfrac{1}{r}\hat{\phi}\\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial\theta} & \dfrac{\partial}{\partial\phi}\\ 0 & 0 & r^2\sin^2\theta\end{vmatrix}\\&=\frac{2\mu_0\sigma R\omega}{3}\left(\cos\theta\,\hat{r}-\sin\theta\,\hat{\theta}\right)\\&=\frac{2}{3}\mu_0\sigma R\omega\hat{z}\end{aligned}$$

The field inside the spherical shell is uniform.

We obtain the magnetic field outside the spherical shell:

$$\vec{B} = \nabla \times \vec{A}$$
$$= \begin{vmatrix} \frac{1}{r^2 \sin\theta}\hat{r} & \frac{1}{r\sin\theta}\hat{\theta} & \frac{1}{r}\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ 0 & 0 & r\sin\theta \frac{\mu_0 \sigma R^4 \omega}{3}\frac{\sin\theta}{r^2} \end{vmatrix}$$
$$= \frac{\mu_0 \sigma R \omega}{3} \begin{vmatrix} \frac{1}{r^2 \sin\theta}\hat{r} & \frac{1}{r\sin\theta}\hat{\theta} & \frac{1}{r}\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ 0 & 0 & \frac{\sin^2\theta}{r} \end{vmatrix}$$
$$= \frac{\mu_0 \sigma R \omega}{3r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)$$

2. Find the vector potential of an infinite solenoid with $n$ turns per unit length, radius $b$, and current $I$.

ANSWER:

Using Stokes' theorem for the vector potential

$$\oint_{\mathcal{C}} \vec{A} \cdot d\vec{s} = \int_{\mathcal{S}} \left(\nabla \times \vec{A}\right) \cdot d\vec{a} = \int_{\mathcal{S}} \vec{B} \cdot d\vec{a} = \Phi$$

where $\Phi$ is the flux of $\vec{B}$ through the loop.

L.H.S.:

$$\oint_{\mathcal{C}} \vec{A} \cdot d\vec{s} = A_\phi \cdot 2\pi r$$

R.H.S.:

$$\begin{cases} \int \vec{B} \cdot d\vec{a} = \mu_0 n I\left(\pi r^2\right), & r \leq b \\ \int \vec{B} \cdot d\vec{a} = \mu_0 n I\left(\pi b^2\right), & r \geq b \end{cases}$$

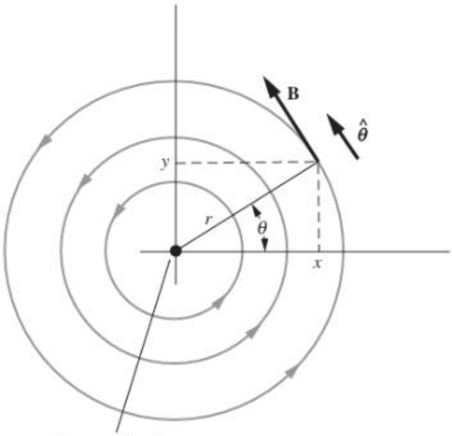
Thus, we obtain

$$\vec{A} = \begin{cases} \frac{\mu_0 n I}{2} r\hat{\phi}, & r \leq b \\ \frac{\mu_0 n I}{2}\frac{b^2}{r}\hat{\phi}, & r \geq b \end{cases}$$

The magnetic field is

$$\vec{B} = \nabla \times \vec{A}$$
$$= \begin{cases} \dfrac{\mu_0 nI}{2} \begin{vmatrix} \frac{1}{r}\hat{r} & \hat{\phi} & \frac{1}{r}\hat{z} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ 0 & r^2 & 0 \end{vmatrix} = \dfrac{\mu_0 nI}{2}\dfrac{1}{r}\dfrac{\partial}{\partial r} r^2 \hat{z} = \mu_0 nI, & r < b \\ \dfrac{\mu_0 nIb^2}{2} \begin{vmatrix} \frac{1}{r}\hat{r} & \hat{\phi} & \frac{1}{r}\hat{z} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ 0 & r\frac{1}{r} & 0 \end{vmatrix} = \dfrac{\mu_0 nIb^2}{2}\dfrac{1}{r}\dfrac{\partial}{\partial r} 1\hat{z} = 0, & r \geq b \end{cases}$$

3. Consider a long straight wire carrying a current $I$.




Outside the wire, what is the vector potential $\vec{A}$?

ANSWER:

$$\vec{B} = \frac{\mu_0 I}{2\pi r}\hat{\theta} = \nabla \times \vec{A} = \begin{vmatrix} \frac{1}{r}\hat{r} & \hat{\theta} & \frac{1}{r}\hat{z} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ A_r & A_\theta & A_z \end{vmatrix} = \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\hat{\theta}$$

Due to the symmetry along the $z$ axis, we can't have any $z$ dependence,

$$\frac{\mu_0 I}{2\pi r}\hat{\theta} = -\frac{\partial A_z}{\partial r}\hat{\theta}$$

$$\Rightarrow A_z = -\int \frac{\mu_0 I}{2\pi r}\, dr = -\frac{\mu_0 I}{2\pi} \ln r$$

$$\Rightarrow \vec{A} = -\frac{\mu_0 I}{2\pi} \ln r\, \hat{z}$$

According to the Helmholtz theorem, $\vec{B}$ is uniquely determined, not $\vec{A}$. $\vec{A}$ could be added any vector function with zero curl. For example,

$$\vec{A} \to \vec{A} = -\frac{\mu_0 I}{2\pi} \ln r\, \hat{z} + \nabla f(\vec{r})$$

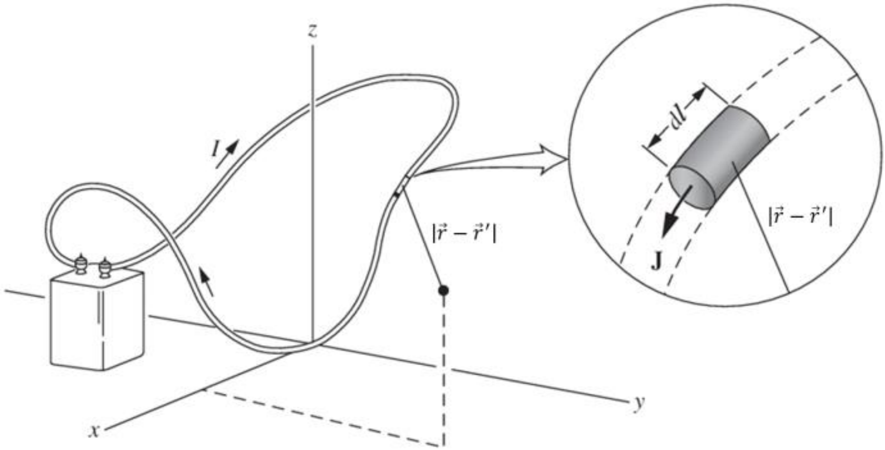
We obtain the magnetic field:

$$\vec{B} = \nabla \times \vec{A} = \nabla \times \left(-\frac{\mu_0 I}{2\pi} \ln r\, \hat{z}\right) + \underbrace{\nabla \times \nabla f}_{=0} = \frac{\mu_0 I}{2\pi r}\, \hat{\theta}$$

# 6-3 Biot-Savart Law

## A. FIELD OF ANY CURRENT-CARRYING WIRE

(1) In many applications we are interested in determining the magnetic field due to a current-carrying circuit. For a thin wire with cross-section area $a$ and current $I$, we have



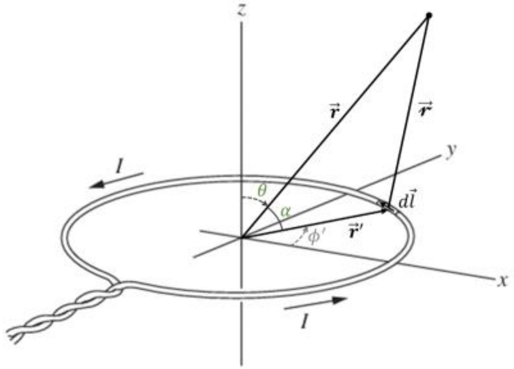

$$J=\frac{I}{a}\Rightarrow Jd^3r=\frac{I}{a}adl=Idl$$

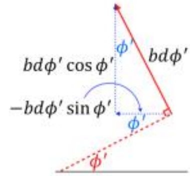
The vector potential at the point becomes

$$\vec{A}=\frac{\mu_0}{4\pi}\int\frac{\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|}d^3r'=\frac{\mu_0 I}{4\pi}\oint_C\frac{d\vec{l}}{|\vec{r}-\vec{r}'|}$$

EXAMPLES:

1. A ring, with radius $b$, lies in the $xy$-plane (centered at the origin) and carries a current $I$ running counterclockwise. Find the vector potential it produces at point $r \gg b$.




ANSWER:

The vector potential of a ring can be written as

$$\vec{A} = \frac{\mu_0 I}{4\pi}\oint_C \frac{1}{\mathscr{r}} d\vec{l}' \text{ where } \mathscr{r} = |\vec{r} - \vec{r}'|$$

$$d\vec{l}' = -bd\phi' \sin\phi' \,\hat{x} + bd\phi' \cos\phi' \,\hat{y}$$

The symmetry of the ring indicates that the contribution of $Id\vec{l}'$ in the $\hat{y}$ direction will cancel each other. Thus, we have

$$\vec{A} = -\frac{\mu_0 I}{4\pi}\int_0^{2\pi} \frac{1}{\mathscr{r}}\hat{x}\sin\phi' \, b\, d\phi' = \frac{\mu_0 I b}{4\pi}\int_0^{2\pi}\frac{\sin\phi'}{\mathscr{r}} d\phi' \hat{\phi}$$

Since $r \gg b$, we have

$$\frac{1}{\mathscr{r}} = \frac{1}{\sqrt{r^2 + b^2 - 2rb\cos\alpha}}$$

$$= \frac{1}{r}\left(1 + \frac{b^2}{r^2} - \frac{2b}{r}\sin\theta\sin\phi'\right)^{-1/2}$$

$$\approx \frac{1}{r}\left(1 - \frac{2b}{r}\sin\theta\sin\phi'\right)^{-1/2}$$

$$\approx \frac{1}{r} + \frac{b}{r^2}\sin\theta\sin\phi' + \cdots$$

The vector potential becomes

$$\vec{A} = \frac{\mu_0 I b}{4\pi}\int_0^{2\pi}\left(\frac{1}{r} + \frac{b}{r^2}\sin\theta\sin\phi'\right)\sin\phi' \, d\phi' \hat{\phi}$$

$$= \frac{\mu_0 I b}{4\pi}\frac{1}{r}\underbrace{\int_0^{2\pi}\sin\phi' \, d\phi'}_{=0} \hat{\phi} + \frac{\mu_0 I b^2}{4\pi}\frac{1}{r^2}\sin\theta\underbrace{\int_0^{2\pi}\sin^2\phi' \, d\phi'}_{=\pi} \hat{\phi}$$

$$= \frac{\mu_0 I b^2}{4}\frac{\sin\theta}{r^2}\hat{\phi}$$

Then, we obtain the magnetic field:

$$\vec{B} = \nabla \times \vec{A}$$
$$= \begin{vmatrix} \frac{1}{r^2 \sin\theta}\hat{r} & \frac{1}{r\sin\theta}\hat{\theta} & \frac{1}{r}\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ 0 & 0 & r\sin\theta\left(\frac{\mu_0 I b^2}{4}\frac{\sin\theta}{r^2}\right) \end{vmatrix}$$
$$= \frac{\mu_0 I b^2}{4}\begin{vmatrix} \frac{1}{r^2 \sin\theta}\hat{r} & \frac{1}{r\sin\theta}\hat{\theta} & \frac{1}{r}\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ 0 & 0 & \frac{\sin^2\theta}{r} \end{vmatrix}$$
$$= \frac{\mu_0 I b^2}{4r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)$$

## B. THE BIOT-SAVART LAW

(1) The magnetic field is then

$$\vec{B} = \nabla \times \vec{A} = \frac{\mu_0 I}{4\pi}\oint_{\mathcal{C}} \nabla_r \times \frac{d\vec{l}'}{\mathscr{r}} = \frac{\mu_0 I}{4\pi}\oint_{\mathcal{C}} \left(\nabla_r \frac{1}{\mathscr{r}}\right) \times d\vec{l}'$$

Since

$$\nabla_r \frac{1}{\mathscr{r}} = -\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$$

we get

$$\vec{B} = \frac{\mu_0 I}{4\pi}\oint_{\mathcal{C}} \left(-\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) \times d\vec{l}' = \frac{\mu_0 I}{4\pi}\oint_{\mathcal{C}} \frac{d\vec{l}' \times \hat{\mathscr{r}}}{\mathscr{r}^2} \cdots\cdots \text{Biot-Savart Law}$$

(2) Sometimes it is convenient to express the equation in two steps

$$\vec{B} = \oint_{\mathcal{C}} d\vec{B}$$

with

$$d\vec{B} = \frac{\mu_0 I}{4\pi}\frac{d\vec{l}' \times \hat{\mathscr{r}}}{\mathscr{r}^2} = \frac{\mu_0 I}{4\pi}\frac{d\vec{l}' \times \vec{\mathscr{r}}}{\mathscr{r}^3}$$

EXAMPLES:

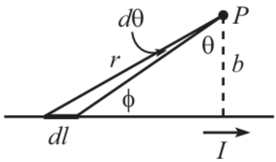
1. Find the magnetic field at a distance $b$ from an infinite straight

wire carrying a steady current $I$.

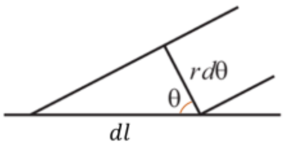
ANSWER:

$$\vec{B} = \frac{\mu_0 I}{4\pi}\int \frac{d\vec{l}\times\hat{r}}{r^2}$$

Consider a small piece of the wire at angle $\theta$, subtending an angle $d\theta$,




If $r$ is the distance from a given point P to the small piece, the length of the piece is




$$dl\cos\theta = rd\theta \Rightarrow dl = \frac{rd\theta}{\cos\theta}$$

Since $b = r\cos\theta$ we obtain

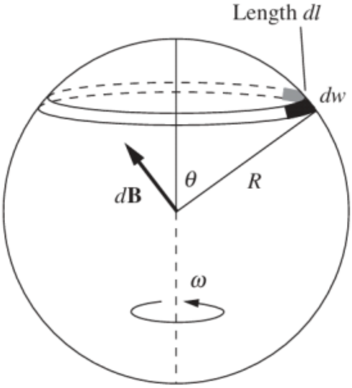
$$dl = \frac{b/\cos\theta\, d\theta}{\cos\theta} = \frac{bd\theta}{\cos^2\theta}$$

The magnetic field is

$$\begin{aligned}
\vec{B} &= \frac{\mu_0 I}{4\pi}\int_{-\pi/2}^{\pi/2}\frac{bd\theta}{\cos^2\theta}\frac{\cos\theta}{(b/\cos\theta)^2}\hat{z} \\
&= \frac{\mu_0 I}{4\pi b}\int_{-\pi/2}^{\pi/2}\cos\theta\, d\theta\hat{z} \\
&= \frac{\mu_0 I}{4\pi b}\sin\theta\Big|_{-\pi/2}^{\pi/2} \\
&= \frac{\mu_0 I}{2\pi b}
\end{aligned}$$

2. A spherical shell with radius $R$ and uniform surface charge den sity $\sigma$ spins with angular frequency $\omega$ around a diameter. Find the magnetic field at the center.

ANSWER:




$$d\vec{B} = \frac{\mu_0 I}{4\pi}\int \frac{d\vec{l}\times\hat{r}}{r^2}$$

The current produced by the ring is

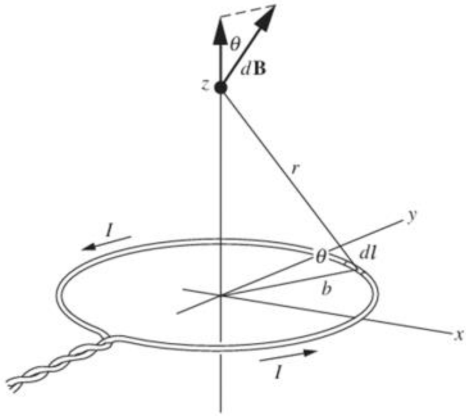
$$I = \frac{dQ}{dt} = \frac{\sigma\big(R\sin\theta\, d\phi\big)(Rd\theta)}{dt} = \sigma R^2\omega\sin\theta\, d\theta$$

$$d\vec{l} = R\sin\theta\, d\phi\hat{\phi}$$

$$\begin{aligned}
d\vec{B} &= \frac{\mu_0 I}{4\pi}\int \frac{d\vec{l}\times\hat{R}}{R^2} \\
&= \frac{\mu_0}{4\pi}\frac{\sigma R^2\omega\sin\theta\, d\theta}{R^2}\int R\sin\theta\, d\phi\,\hat{\phi}\times\hat{R} \\
&= \frac{\mu_0\sigma R\omega\sin^2\theta\, d\theta}{4\pi}2\pi\hat{\theta} \\
&= \frac{\mu_0\sigma R\omega\sin^2\theta\, d\theta}{2}\hat{\theta}
\end{aligned}$$

$$\begin{aligned}
\vec{B}_z &= \int d\vec{B}\sin\theta \\
&= \frac{\mu_0\sigma R\omega}{2}\int_0^{\pi}\sin^3\theta\, d\theta\hat{z} \\
&= \frac{\mu_0\sigma R\omega}{2}\left(-\cos\theta + \frac{\cos^3\theta}{3}\right)_0^{\pi}\hat{z} \\
&= \frac{2}{3}\mu_0\sigma R\omega\hat{z}
\end{aligned}$$

3. Find the magnetic field at a point on the axis of a ring of radius $b$ that carries a current $I$.




ANSWER:

- Method I:

$$d\vec{B} = \frac{\mu_0 I}{4\pi}\frac{d\vec{l} \times \hat{r}}{r^2}$$

Consider the field on the $z$-axis:

Each element of the ring of length $d\vec{l}$ contributes a $d\vec{B}$ perpendicular to $\vec{r}$. The total field on the axis must point in the $z$ direction,

$$dB_z = d\vec{B}\cos\theta = \frac{\mu_0 I}{4\pi}\frac{dl}{r^2}\cos\theta\,\hat{z}$$

Since

$$b = r\cos\theta$$

we obtain

$$dB_z = \frac{\mu_0 I}{4\pi}\frac{dl}{r^2}\frac{b}{r}\hat{z} = \frac{\mu_0 I}{4\pi}\frac{b}{r^3}dl\hat{z}$$

So the field on the axis at any point $z$ is

$$B_z = \frac{\mu_0 I}{4\pi}\frac{b}{r^3}\int dl\,\hat{z} = \frac{\mu_0 I}{4\pi}\frac{b}{r^3}2\pi b = \frac{\mu_0 I b^2}{2r^3}$$

- Method II:

The vector potential at any point is

$$\vec{A} = \frac{\mu_0 I b^2}{4}\frac{\sin\theta}{r^2}\hat{\phi}$$

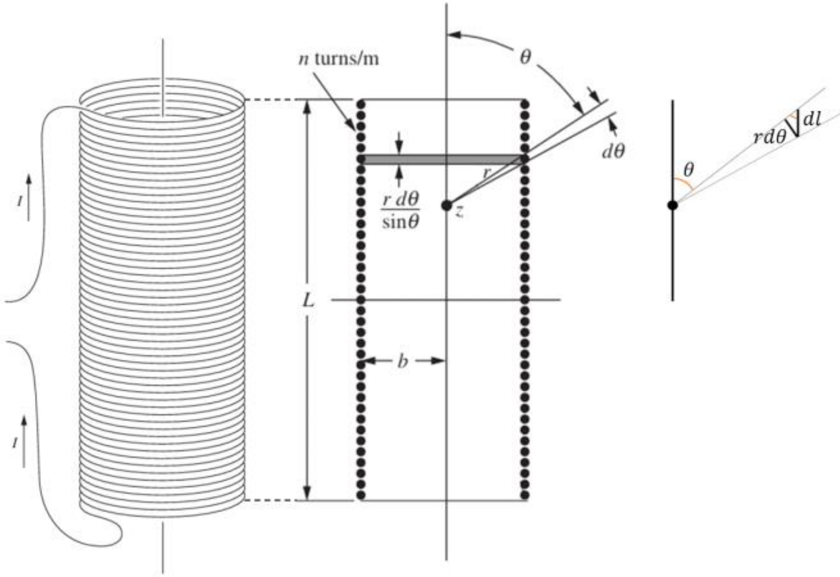
The magnetic at any point is

$$\vec{B} = \nabla \times \vec{A} = \frac{\mu_0 I b^2}{4r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)$$

The magnetic field on the $z$-axis, i.e., $\theta = 0$:

$$\vec{B} = \frac{\mu_0 I b^2}{2r^3}\hat{z}$$

4. The magnetic field of an infinite solenoid of current with radius $b$




Consider the contribution from the current rings included between radii from the point $z$ that make angles $\theta$ and $\theta + d\theta$ with the axis.

$$dB_z = \frac{\mu_0 I b^2}{2r^3} n dl$$

The length of this segment of the solenoid is

$$dl \sin\theta = r d\theta \Rightarrow dl = \frac{r d\theta}{\sin\theta}$$

Thus, we obtain

$$dB_z = \frac{\mu_0 I b^2}{2r^3} n \frac{r d\theta}{\sin\theta} = \frac{\mu_0 n I b^2 d\theta}{2r^2 \sin\theta}$$

Since

$$b = r\sin\theta$$

we obtain

$$dB_z = \frac{n I r d\theta}{\sin\theta}\frac{\mu_0 r^2 \sin^2\theta}{2r^3} = \frac{\mu_0 n I}{2}\sin\theta\, d\theta$$

For the infinite solenoid, we obtain

$$B_z = \int dB_z = \int_0^{\pi} \frac{\mu_0 n I}{2}\sin\theta\, d\theta = \frac{\mu_0 n I}{2}\cos\theta\Big|_0^{\pi} = \mu_0 n I$$